Mini-course on GFD

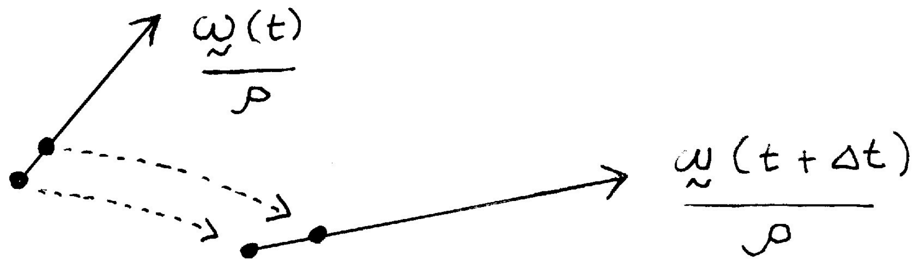
What *is* GFD?

The fluid dynamics of potential vorticity.

$$\mathbf{v}(\mathbf{x},t) = \text{velocity of the fluid at location } \mathbf{x} \text{ and time } t$$

$$\omega(\mathbf{x},t) \equiv \nabla \times \mathbf{v}$$

homentropic case : $p = p(\rho)$ instead of $p = p(\rho, S)$

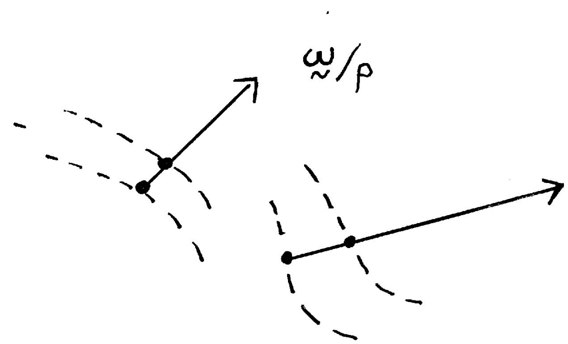


$$\frac{\partial}{\partial t}\left(\frac{\omega_i}{\rho}\right) = \left(\frac{\omega_j}{\rho}\right)\frac{\partial}{\partial x_j} v_i - v_j \frac{\partial v}{\partial x_j}\left(\frac{\omega_i}{\rho}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\omega}{\rho}\right) = -L_{\mathbf{v}}\left(\frac{\omega}{\rho}\right)$$

introduce: $\theta(\mathbf{x},t)$

$$\frac{D\theta}{Dt}=0$$



The flow affects $\nabla\theta$ in the opposite way from ω/ρ

$$\frac{\partial}{\partial t}\nabla\theta=-L_{\mathbf{v}}\nabla\theta$$

$$\frac{\partial}{\partial t}\left(\frac{\omega}{\rho}\right) = -L_{\mathbf{v}}\left(\frac{\omega}{\rho}\right)$$

$$\frac{\partial}{\partial t}\nabla\theta = -L_{\mathbf{v}}\nabla\theta$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}\left(\frac{\omega}{\rho}\cdot\nabla\theta\right) = -L_{\mathbf{v}}\left(\frac{\omega}{\rho}\cdot\nabla\theta\right)$$

$$\Leftrightarrow \qquad \frac{D}{Dt}\left(\frac{\omega}{\rho}\cdot\nabla\theta\right) = 0$$

(Ertel’s theorem, homentropic case)

Ertel (1942), Cauchy (1827)

$$\frac{D}{Dt}\left(\frac{\omega}{\rho}\cdot\nabla\theta\right) \equiv \frac{D}{Dt}Q = 0$$

$$\frac{D\theta_1}{Dt} = 0, \quad \frac{D\theta_2}{Dt} = 0, \quad \frac{D\theta_3}{Dt} = 0$$ (Lagrangian coordinates)

$$\frac{DQ_1}{Dt} = 0, \quad \frac{DQ_2}{Dt} = 0, \quad \frac{DQ_3}{Dt} = 0$$

$$\mathbf{Q} \equiv (Q_1, Q_2, Q_3) = \left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_3}\right) \times (A_1, A_2, A_3)$$

where $\mathbf{v} = A_1\nabla\theta_1 + A_2\nabla\theta_2 + A_3\nabla\theta_3$

provided $d\theta_1\, d\theta_2\, d\theta_3 = d(mass)$

Summary of the homentropic case

(x, y, z, t) coordinates

$(\theta_1, \theta_2, \theta_3, \tau)$ coordinates

$$\mathbf{v} = u\nabla x + v\nabla y + w\nabla z$$

$$\mathbf{v} = A_1\nabla\theta_1 + A_2\nabla\theta_2 + A_3\nabla\theta_3$$

$$(\omega_1, \omega_2, \omega_3) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (u, v, w)$$

$$(Q_1, Q_2, Q_3) = \left(\frac{\omega}{\rho}\cdot\nabla\theta_1, \frac{\omega}{\rho}\cdot\nabla\theta_2, \frac{\omega}{\rho}\cdot\nabla\theta_3\right) = \left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_3}\right) \times (A_1, A_2, A_3)$$

$$\frac{\partial}{\partial t}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho}\cdot\nabla\right)\mathbf{v} - (\mathbf{v}\cdot\nabla)\frac{\omega}{\rho}$$

$$\frac{\partial}{\partial\tau}\mathbf{Q} = 0$$

However, the *simplicity* of the Lagrangian description is offset by the *complexity* of the transformation between coordinate systems.

General, nonhomentropic, case: $p = p(\rho, S)$

The presence of entropy gradients destroys 2/3 of the conservation law,

$$D\mathbf{Q} / Dt = 0$$

but endows the surviving 1/3 with a physical importance that does not decrease as the Lagrangian coordinates become ever more convoluted.

This is the essence of GFD.

when entropy gradients are present…. *pressure torque* appears

$$\frac{\partial}{\partial t}\left(\frac{\omega}{\rho}\right) = -L_{\mathrm{v}}\left(\frac{\omega}{\rho}\right) + \frac{\nabla\rho \times \nabla p(\rho,S)}{\rho^3}$$

$$\frac{DQ_i}{Dt} = \frac{\nabla\theta_i \cdot \left(\nabla\rho \times \nabla p(\rho,S)\right)}{\rho^3}$$

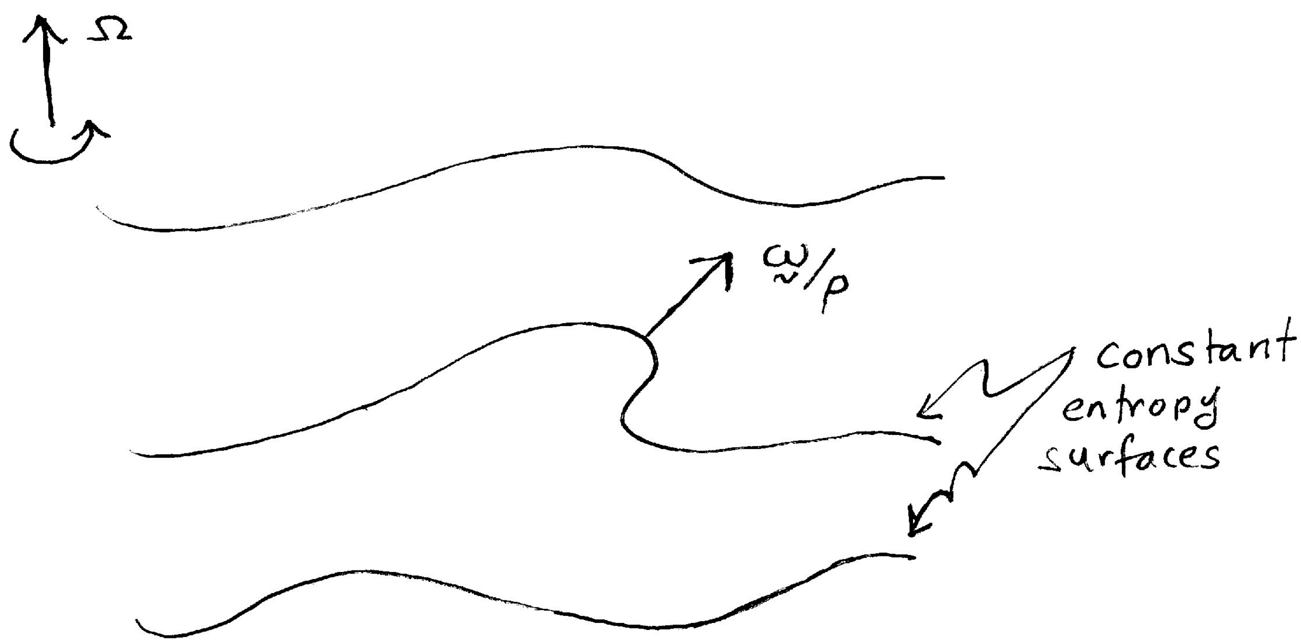
Choose one of the θ_i to be S itself…

$$\frac{D}{Dt}\left(\frac{\omega \cdot \nabla S}{\rho}\right) = 0$$

The other two conservation laws are destroyed.

However, the surviving conservation law is more useful than before because gyroscopic forces and gravitational restoring forces resist the folding of entropy surfaces.

Gyroscopic or gravitational restoring forces resist the irreversible folding of constant-entropy surfaces.



Motivated approach: Hamilton's principle for a perfect fluid

$$\int L\,d\boldsymbol{\tau} = \int d\boldsymbol{\tau} \iiint \underbrace{da\,db\,dc}_{d\theta_1\,d\theta_2\,d\theta_3} \left[\frac{1}{2} \frac{\partial \mathbf{x}}{\partial \boldsymbol{\tau}} \cdot \frac{\partial \mathbf{x}}{\partial \boldsymbol{\tau}} - E\big(\alpha, S(a,b,c)\big) - \Phi(x,y,z) \right] = 0$$

$$\alpha = \frac{\partial(x,y,z)}{\partial(a,b,c)}$$

$$\delta x(a,b,c,\boldsymbol{\tau}), \delta y(a,b,c,\boldsymbol{\tau}), \delta z(a,b,c,\boldsymbol{\tau}) \quad \Rightarrow \quad \text{fluid equations}$$

Eulerian version: $\delta a(x,y,z,t), \delta b(x,y,z,t), \delta c(x,y,z,t)$

homentropic case:

$$\delta \frac{\partial(a,b,c)}{\partial(x,y,z)} = 0 \quad \Rightarrow \quad \delta(a,b,c) = \left(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c} \right) \times \delta \mathbf{T}(a,b,c,\tau)$$

$$\Rightarrow \quad \delta \int L\,d\tau = \int d\tau \iiint d\mathbf{a}\ \frac{\partial}{\partial \boldsymbol{\tau}} \left[\nabla_{\mathbf{a}} \times \mathbf{A} \right] \cdot \delta \mathbf{T} = 0$$

if entropy gradients are present:

$$\delta \frac{\partial(a,b,S)}{\partial(x,y,z)} = 0 \ \text{ and } \ \delta S = 0 \quad \Rightarrow \quad \delta a = -\frac{\partial}{\partial b} \delta \psi, \quad \delta b = \frac{\partial}{\partial a} \delta \psi$$

$$\Rightarrow \quad \frac{\partial}{\partial \boldsymbol{\tau}} \left[\left(\nabla_{\mathbf{a}} \times \mathbf{A} \right) \cdot \nabla_{\mathbf{a}} S \right] = 0$$

The particle-relabeling symmetry is the essence of what it means to be a fluid.

$$\int L\,d\tau = \int d\tau \iiint da\,db\,dc \left[\frac{1}{2}\frac{\partial \mathbf{x}}{\partial \tau}\cdot\frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(x,y,z)}{\partial(a,b,c)}, S(a,b,c)\right) - \Phi(x,y,z)\right]$$

Potential vorticity is unique to fluid mechanics.

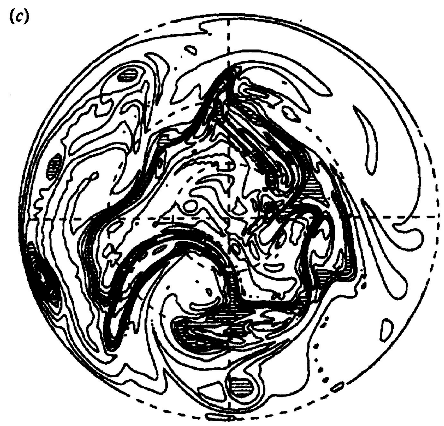
Approximations that do not respect potential vorticity conservation must be viewed with suspicion.

Potential vorticity was first discovered (and has proved most useful) in the study of rotating and/or stratified fluids (GFD) but it is likely to be significant in the study of turbulence.

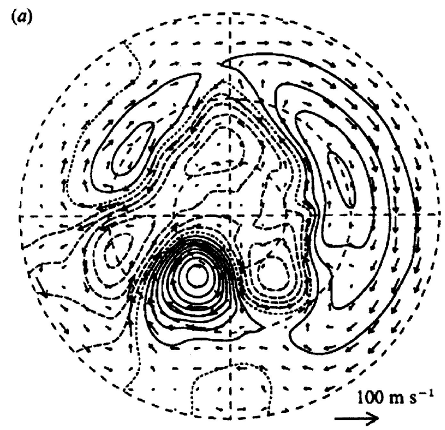
Within GFD: the “2+1 view” prevails

M. E. McIntyre & W. A. Norton (1988-2000):
“Potential vorticity inversion on a hemisphere”

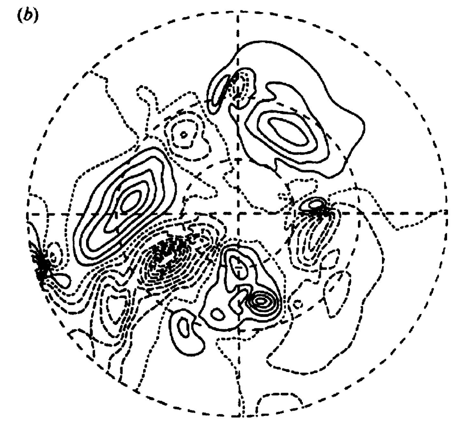
Dynamical evolution using shallow-water dynamics:



potential vorticity



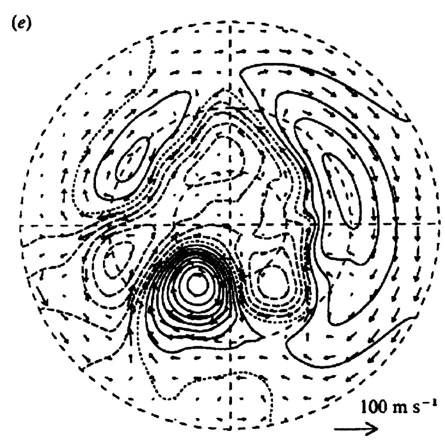
fluid depth

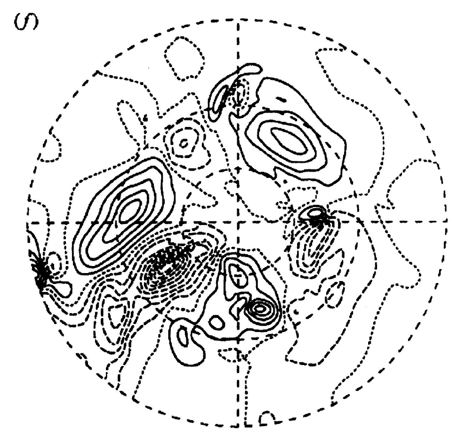


divergence

Reconstruction from the PV
at a fixed time:

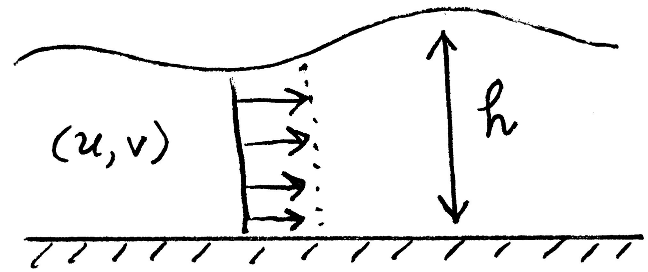
“2+1 view of GFD”





Also, more recent work by Ali Mohebalhojeh & D. Dritschel

Shallow-water equations



$$\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\Rightarrow \frac{Dq}{Dt} = 0$$

$$u(x,y,t),\ v(x,y,t),\ h(x,y,t)$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$q = \frac{\zeta + f}{h}$$

SWE are the prototype for the 3d *primitive equations*
= general fluid eqns + hydrostatic approx + Boussinesq approx
+traditional approx

in (x, y, S, t) coordinates: $\frac{D}{Dt}\left(\frac{\zeta + f}{\partial z / \partial S}\right)$

"2+1 view" of the SWE: linearized dynamics

linear dynamics with $f = f_0$

$$\frac{\partial u}{\partial t} - f_0 v = -g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + f_0 u = -g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + H_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

waves $\propto \exp(ikx + ily - i\omega t)$

$$\omega = 0 \text{ or } \omega^2 = f_0{}^2 + gH_0\left(k^2 + l^2\right)$$

Normal-mode variables:

$$q = \zeta - \frac{f_0}{H_0}h, \quad \delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad \zeta_{AG} = \zeta - \frac{g}{f_0}\nabla^2 h$$

$\Leftrightarrow$

$$\frac{\partial q}{\partial t} = 0$$

$$\frac{\partial \delta}{\partial t} - f_0\,\zeta_{AG} = 0$$

$$\frac{\partial \zeta_{AG}}{\partial t} + f_0\,\delta - \frac{gH_0}{f_0}\nabla^2\delta = 0$$

Potential vorticity inversion at a fixed time:

take $\delta = \zeta_{AG} = 0$

solve $\frac{g}{f_0}\nabla^2 h - \frac{f_0}{H_0}h = q$ for h

then $u = -\frac{g}{f_0}\frac{\partial h}{\partial y}, \quad v = \frac{g}{f_0}\frac{\partial h}{\partial x}$

Nonlinear SWE with variable f and H

$$\frac{\partial q}{\partial t} = \{q,q\} + \{q,f'\} + \{q,H'\} + \{q,\delta\} + \{q,\zeta_{AG}\}$$

$$\frac{\partial \delta}{\partial t} - f_0\,\zeta_{AG} = \{\ ,\ \}$$

$$\frac{\partial \zeta_{AG}}{\partial t} + f_0\,\delta - \frac{gH_0}{f_0}\nabla^2\delta = \{\ ,\ \}$$

Set $\delta = \zeta_{AG} = 0$ in the first eqn, and throw away the others.

This is a Galerkin approximation. It is also a (metric) projection.

the result:
$$\frac{\partial}{\partial t}\left(\frac{g}{f_0}\nabla^2 h - \frac{f_0}{H_0}h\right) = -\left(-\frac{g}{f_0}\frac{\partial h}{\partial y}, \frac{g}{f_0}\frac{\partial h}{\partial x}\right)\cdot\nabla\left(\frac{g}{f_0}\nabla^2 h - \frac{f_0}{H_0}h + f + \frac{f_0}{H_0}(H_0 - H)\right)$$

$$\Leftrightarrow\quad \frac{\partial q}{\partial t} + \frac{\partial(\psi,q)}{\partial(x,y)} = 0, \qquad q = \nabla^2\psi - \frac{f_0^{\,2}}{gH_0}\psi + f + \frac{f_0}{H_0}(H_0 - H) \quad \text{where } \psi \equiv \frac{gh}{f_0}$$

Single-layer QG dynamics

$$\frac{\partial q}{\partial t} + \frac{\partial(\psi,q)}{\partial(x,y)} = 0, \qquad q = \nabla^2\psi - \frac{f_0^{\,2}}{gH_0}\psi + f + \frac{f_0}{H_0}(H_0 - H)$$

QG is the simplest useful balance model.

$$\left(h = \frac{f_0}{g}\psi, \quad u = -\frac{\partial\psi}{\partial y}, \quad v = +\frac{\partial\psi}{\partial x}\right)$$

Useful extensions: multi-layer flows, continuously stratified flow.

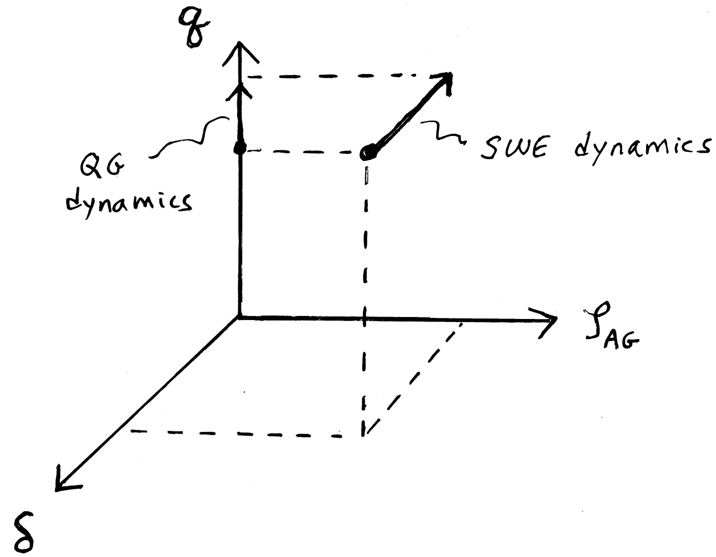
Early use in numerical weather prediction.

Biggest defect of QG: it is closely tied to a reference state.

Leading-order alternatives to QG avoid this at a cost in complexity.

Higher-order balance models.

QG is a metric projection onto the slow manifold



but phase space has no metric!

Hamiltonian fluid dynamics offers another kind of projection:

1. Hamilton's principle + constraints
2. Restriction of a differential form
3. Dirac bracket

$\Longrightarrow$ Semigeostrophic equations (SG)

SG, and an abstract view of QG

The state of the fluid corresponds to a point in phase space:

$$z = \left(z^1, z^2, z^3 \ldots\right)$$

Exact dynamics in Hamiltonian form: $$\frac{d}{dt}F(z) = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial H}{\partial z^i}$$

QG dynamics: $$\frac{d}{dt}F(z) = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial H}{\partial z^i} + \mu_{(m)} \frac{\partial F}{\partial z^i} g^{ij} \frac{\partial \mu_{(m)}}{\partial z^i}$$

SG dynamics: $$\frac{d}{dt}F(z) = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial H}{\partial z^i} + \mu_{(m)} \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial \mu_{(m)}}{\partial z^i}$$

SG corresponds to attaching constraints to Hamilton's principle.

Unlike QG, SG is *not* tied to a particular reference state,
but it is much harder to solve than QG.

Single-layer QG

$$\frac{\partial q}{\partial t} + \frac{\partial(\psi,q)}{\partial(x,y)} = 0, \qquad q = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{f_0^{\,2}}{gH_0}\psi + f + \frac{f_0}{H_0}(H_0 - H)$$

3d QG

$$\frac{\partial q}{\partial t} + \frac{\partial(\psi,q)}{\partial(x,y)} = 0, \qquad q = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^{\,2}}{N(z)^2}\frac{\partial\psi}{\partial z}\right) + f$$

top/bottom boundary condition $$\left[\frac{\partial}{\partial t} + \frac{\partial(\psi,\bullet)}{\partial(x,y)}\right]\left(\frac{\partial\psi}{\partial z}\right) + \frac{N(z)^2}{f_0}w = 0$$

Simplest case:

1. constant f
2. flat top & bottom boundaries
3. uniform $N(z)$

Simplest-case QG

Single layer: $$\left[\frac{\partial}{\partial t}+\frac{\partial(\psi,\bullet)}{\partial(x,y)}\right]\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0 \quad \Leftrightarrow \quad \text{2d Euler dynamics}$$

3d: $$\left[\frac{\partial}{\partial t}+\frac{\partial(\psi,\bullet)}{\partial(x,y)}\right]\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)=0$$

VERSUS:

3d Euler dynamics: $$\frac{\partial}{\partial t}\omega_i+v_j\frac{\partial}{\partial x_j}\omega_i-\omega_j\frac{\partial}{\partial x_j}v_i=0, \qquad \omega_i=\varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

Euler dynamics + viscosity = Navier-Stokes dynamics

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v}, \qquad \nabla\cdot\mathbf{v} = 0$$

2d Navier-Stokes $\Leftrightarrow$ Single-layer QG with no accessories

Essence of GFD is the huge difference between
2d and 3d Navier-Stokes

Conservation laws are the key.

In the limit $\nu \to 0$, 3d Euler conserves $E = \iiint d\mathbf{x}\ \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$

2d Euler conserves $E = \iint d\mathbf{x}\ \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = \iint d\mathbf{x}\ \frac{1}{2}\nabla\psi\cdot\nabla\psi$

and $Z = \iint d\mathbf{x}\ \frac{1}{2}\omega\cdot\omega = \iint d\mathbf{x}\ \frac{1}{2}\left(\nabla^2\psi\right)^2$

Navier-Stokes dynamics

In 2d and in 3d, $\varepsilon \equiv \frac{dE}{dt} = -\nu \iiint d\mathbf{x}\, \omega \cdot \omega \equiv -2\nu\, Z$

In 3d, Z is unbounded. $\left(\varepsilon \sim U^3 / L\right.$ is independent of $\left.\nu.\right)$

Vortex stretching transfers energy to arbitrarily small scales in a finite time.

In 2d, Z can only decrease. The impossibility of irreversible vortex stretching traps the energy in large scales.

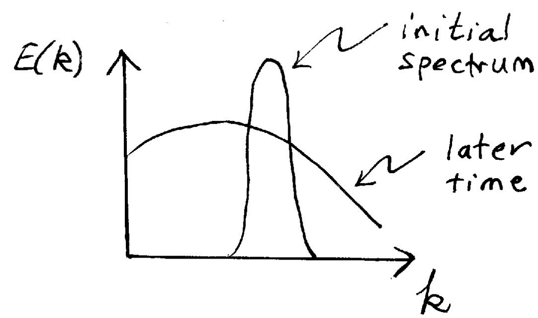
2d inviscid Euler

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \zeta = 0 \qquad \Leftrightarrow \qquad \frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x,y)} = 0$$

$$\Rightarrow \qquad \frac{dE}{dt} = \frac{dZ}{dt} = 0$$

$$E = \frac{1}{2} \iint dx\, dy\ \ \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} \iint dx\, dy\ \ \nabla \psi \cdot \nabla \psi = \int_0^\infty dk\ \ E(k)$$

$$Z = \frac{1}{2} \iint dx\, dy\ \ \zeta^2 = \frac{1}{2} \iint dx\, dy\ \ \left(\nabla^2 \psi\right)^2 = \int_0^\infty dk\ \ k^2 E(k)$$



3d inviscid QG

$$\frac{\partial q}{\partial t} + \frac{\partial(\psi, q)}{\partial(x, y)} = 0, \qquad q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{f_0^{\ 2}}{N_0^{\ 2}} \frac{\partial^2 \psi}{\partial z^2}$$

$$\Rightarrow \qquad \frac{dE}{dt} = \frac{dZ}{dt} = 0$$

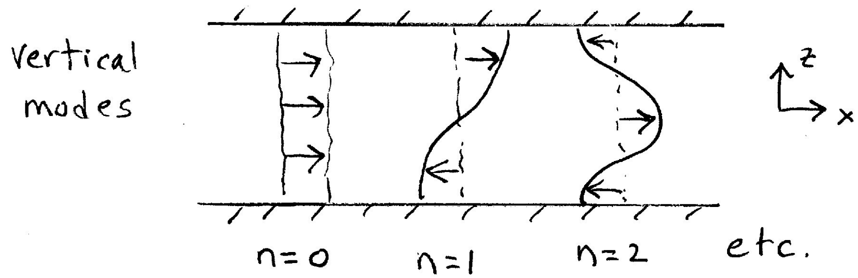
$$E = \frac{1}{2} \iiint dx\, dy\, dz \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 + \frac{f_0^{\ 2}}{N_0^{\ 2}} \left(\frac{\partial \psi}{\partial z} \right)^2 \right] = \int_0^\infty dk_H \int_0^\infty dk_V \;\; E(k_H, k_V)$$

$$Z = \frac{1}{2} \iiint dx\, dy\, dz \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{f_0^{\ 2}}{N_0^{\ 2}} \frac{\partial^2 \psi}{\partial z^2} \right]^2 = \int_0^\infty dk_H \int_0^\infty dk_V \;\; k_{total}^{\ 2} \, E(k_H, k_V)$$

Energy transfer is to low $\quad k_{total} \equiv \sqrt{k_x^{\ 2} + k_y^{\ 2} + \frac{f_0^{\ 2}}{N_0^{\ 2}} k_z^{\ 2}}$

3d inviscid QG: vertical modes

$$\frac{\partial q}{\partial t} + \frac{\partial(\psi, q)}{\partial(x, y)} = 0, \qquad q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{f_0^{\ 2}}{N_0^{\ 2}} \frac{\partial^2 \psi}{\partial z^2}$$



Energy transfer is to low $k_{total} \equiv \sqrt{k_x^{\ 2} + k_y^{\ 2} + k_n^{\ 2}}$

$$k_n = \frac{f_0}{N_0} \frac{n\pi}{H_0} = \frac{1}{n\text{ - th deformation radius}}$$

Flow becomes depth-invariant at horizontal scales > $1/k_1$

Ocean currents observed south of the separated Gulf Stream

(Peter Rhines & Bill Schmitz)

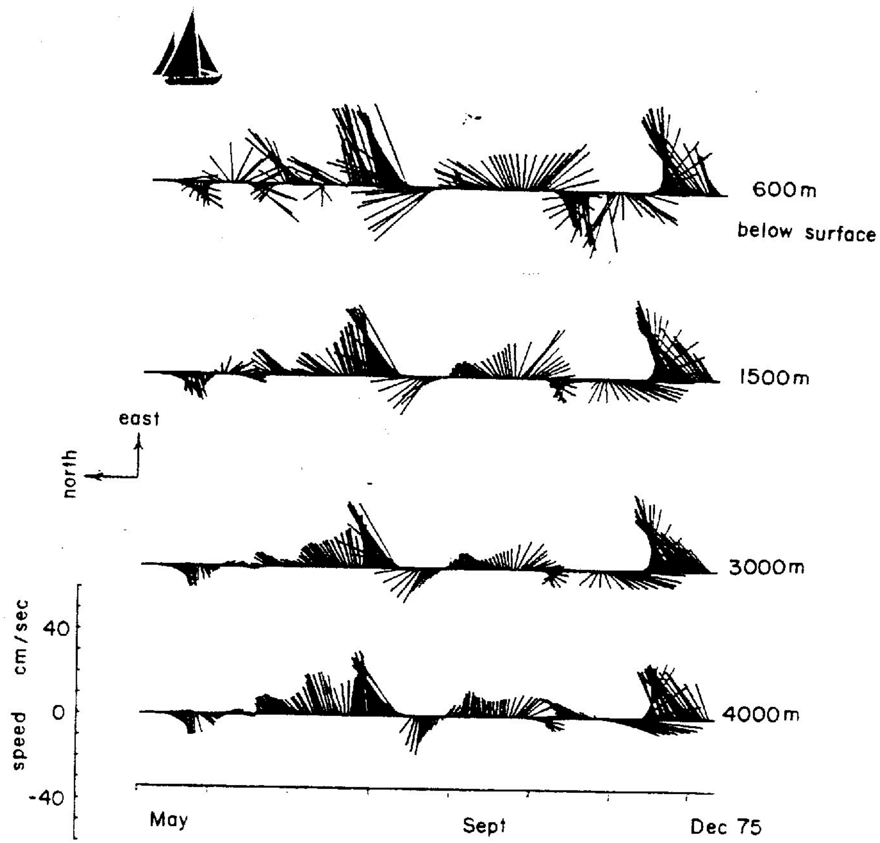


Figure 12b Currents observed at a more energetic part of the North Atlantic, just south of the Gulf Stream (37° 30′N, 55° W). Note different scales from Figure 12a. Here, the currents exhibit 50-day periods, roughly, and have penetrated right to the ocean bottom. Strong, depth-independent turbulence is characteristic of the analagous models. (See Schmitz 1978.)

3d inviscid QG: WKB form

$$\frac{\partial q}{\partial t} + \frac{\partial(\psi, q)}{\partial(x, y)} = 0, \qquad q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{(\beta y)^2}{N_0^{\,2}} \frac{\partial^2 \psi}{\partial z^2}$$

Energy transfer is to low $\quad k_{total} \equiv \sqrt{k_x^{\,2} + k_y^{\,2} + k_n(y)^2}$

$$k_n(y) = \frac{\beta y}{N_0} \frac{n\pi}{H_0} = \frac{1}{n\text{ - th deformation radius}}$$

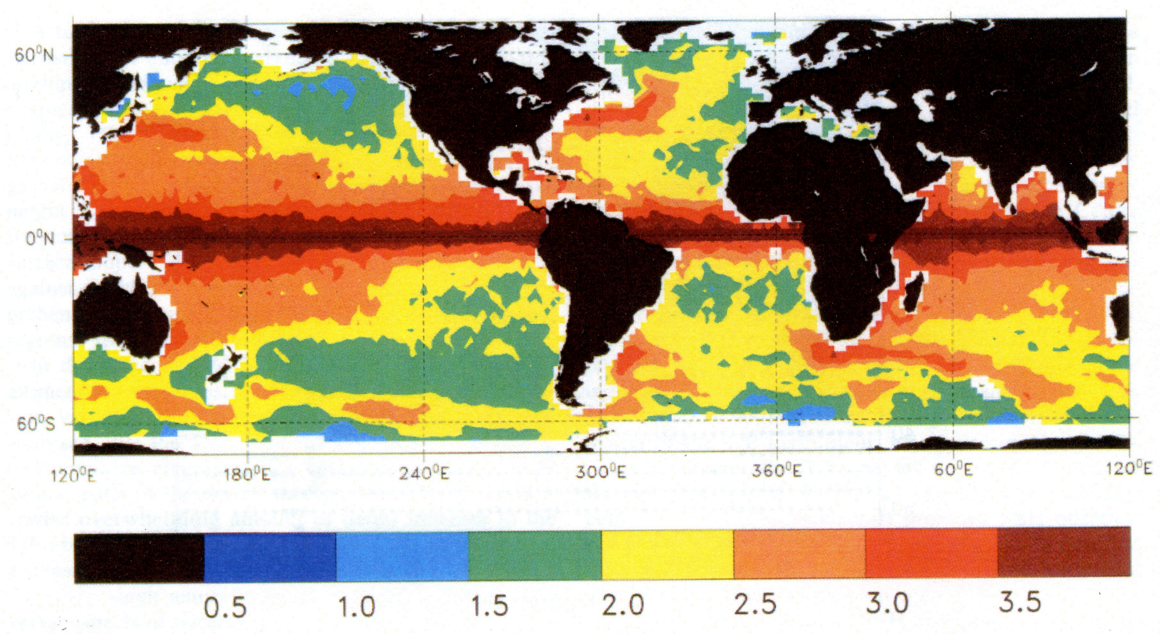
Energy transfer is *toward* the equator and into *high vertical mode*.

The energy in mode n shows an equatorial peak of width W, determined by $k_y = k_n$ i.e.

$$\frac{1}{W_n} = \frac{\beta W_n}{N_0} \frac{n\pi}{H_0} \quad \Rightarrow \quad W_n = \sqrt{\frac{N_0 H_0}{\beta\, n\pi}} \equiv \text{equatorial deformation radius}$$

“Global characteristics of ocean variability estimated from regional TOPEX/POSEIDON altimeter measurements”

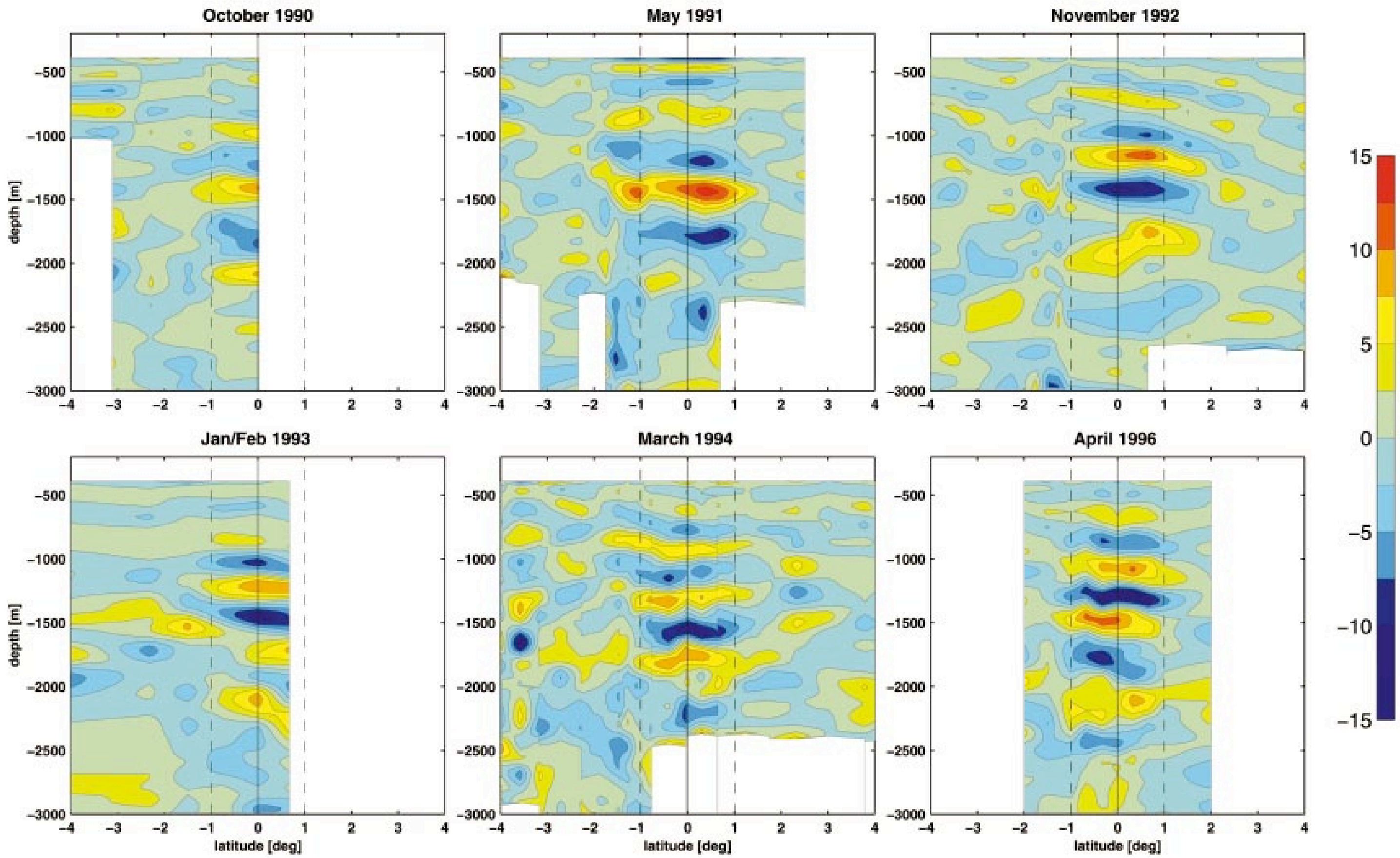
D. Stammer, *J. Phys. Oc.* 1997



Eddy kinetic energy at the sea surface (cm/sec)**2

U. Send, C. Eden & F. Schott

"Atlantic equatorial deep jets…" *J. Phys. Oc.* 2002



Zonal flow (cm/sec) along the equator from six cruises.

Rectified flow

$$\frac{\partial q}{\partial t}+\frac{\partial(\psi,q)}{\partial(x,y)}=0, \qquad q=\underbrace{\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}}_{\zeta}+\underbrace{f+\frac{f_0}{H_0}(H_0-H)}_{h}$$

Conserved quantities are

$$E=\iint d\mathbf{x}\ \frac{1}{2}\nabla\psi\cdot\nabla\psi \quad \text{and} \quad Z=\iint d\mathbf{x}\ \frac{1}{2}q^2=\iint d\mathbf{x}\ \frac{1}{2}\zeta^2+\iint d\mathbf{x}\,\zeta\,h+const$$

The system seeks the state of energy equipartition. If $h=0$, the conservation of $\iint d\mathbf{x}\ \zeta^2$ prevents this.

However, if $h\neq 0$ then $\iint d\mathbf{x}\ \zeta^2$ can increase if $\iint d\mathbf{x}\,\zeta\,h$ becomes negative.

Anticyclonic flow over seamounts

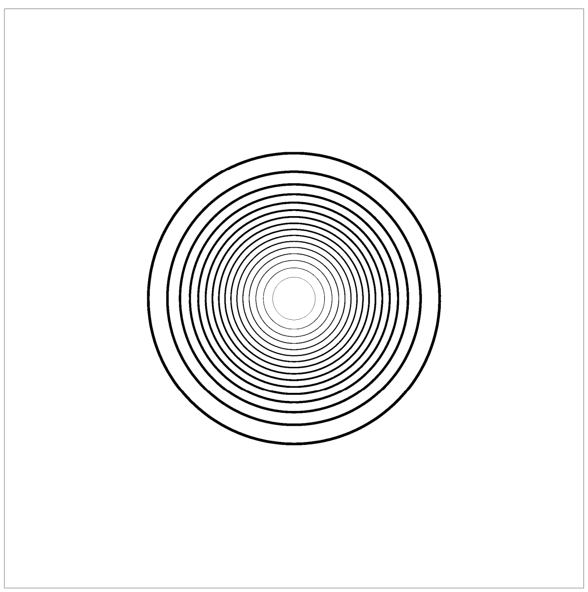
Numerical example: Seamount in a square ocean

Seamount 1 km high in an ocean 4 km deep and 4000 km wide

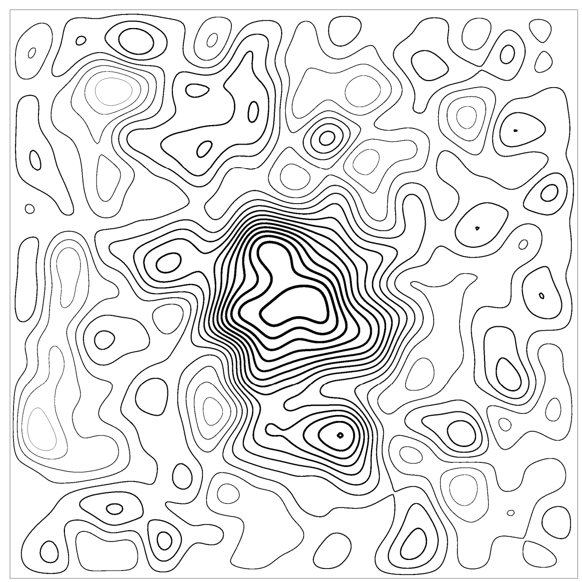
$$f = 2\pi / day$$

Navier-Stokes friction

Random initial conditions with rms velocity = 50 km/day

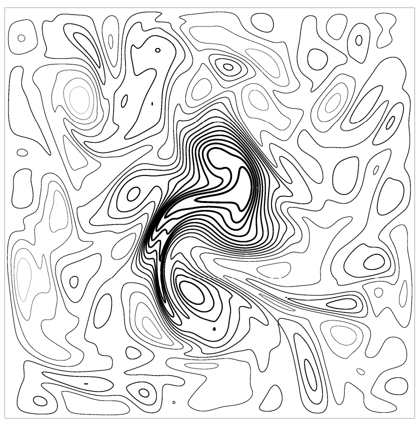


ocean depth

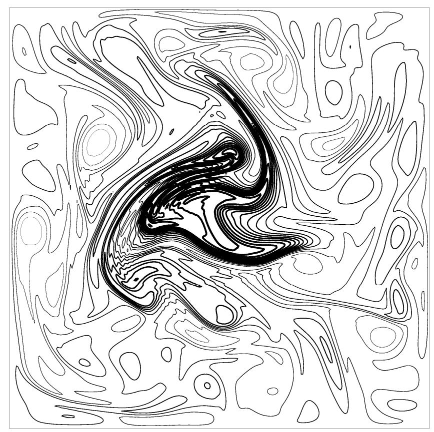


initial potential vorticity

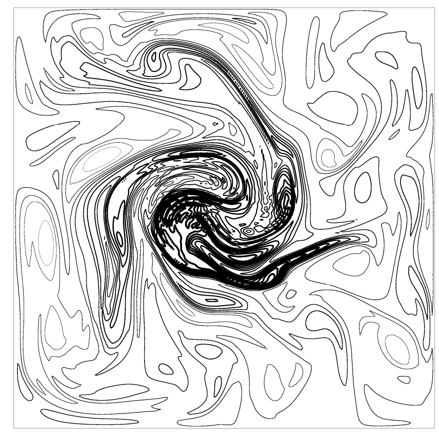
Potential vorticity at subsequent times



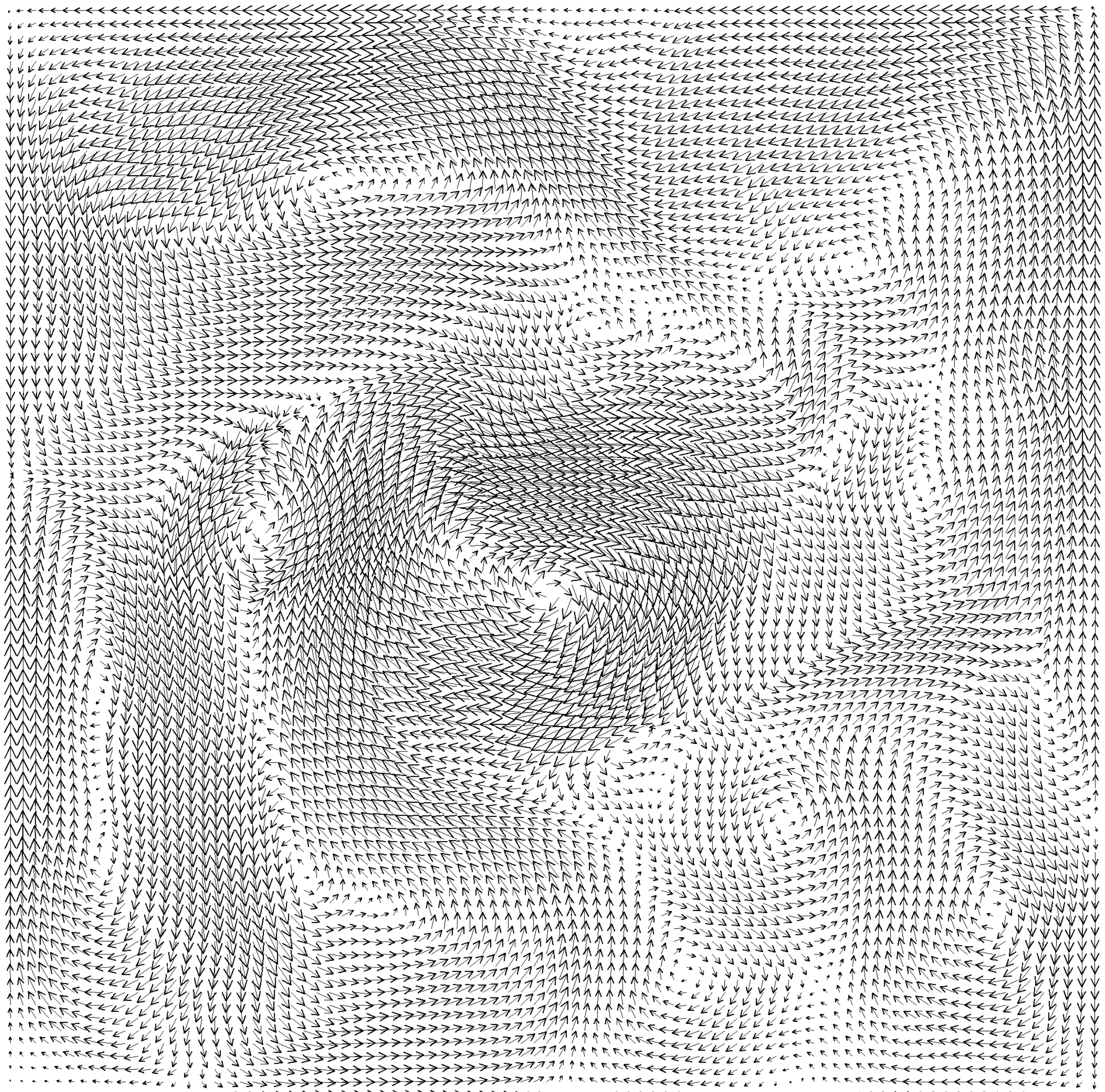
6 days



19 days

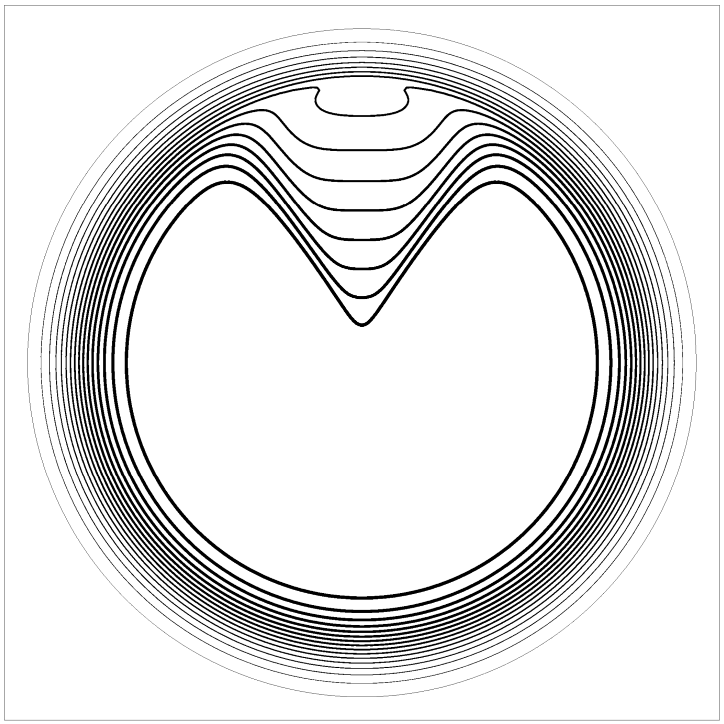


31 days

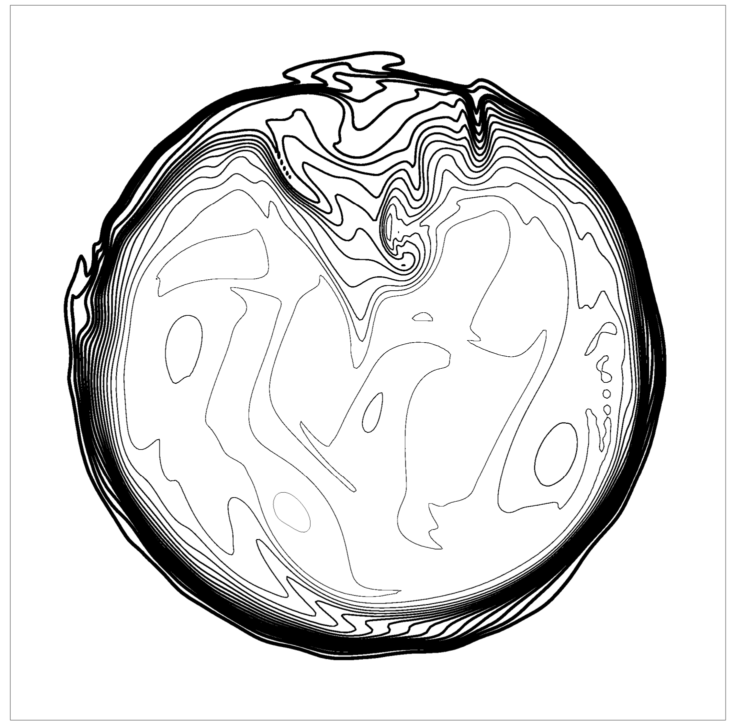


Final velocity

Ocean basin with a protruding ridge

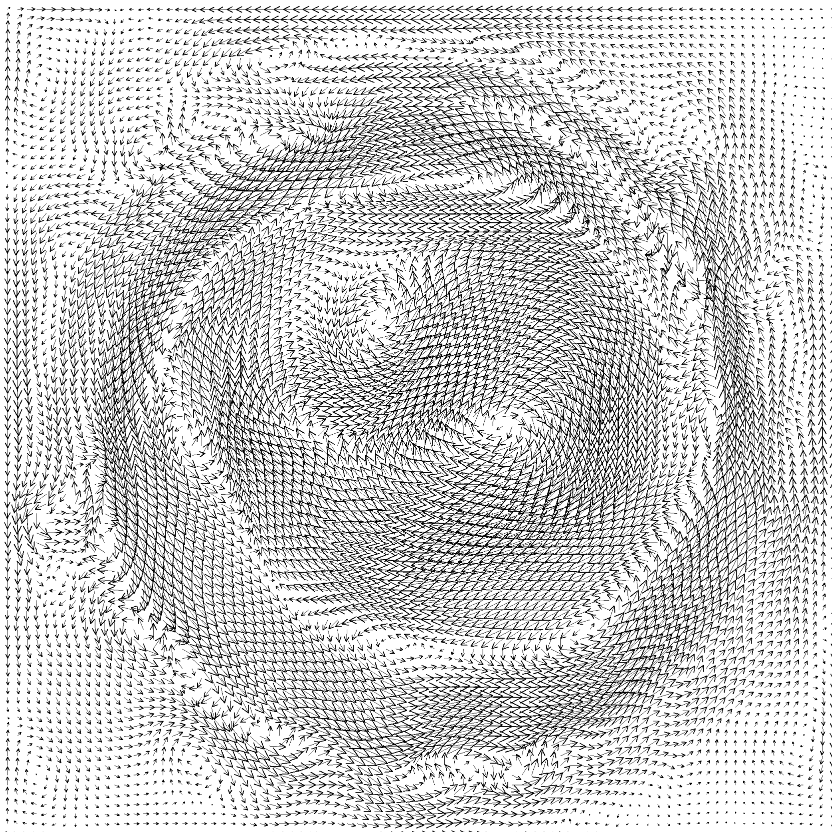


Ocean depth

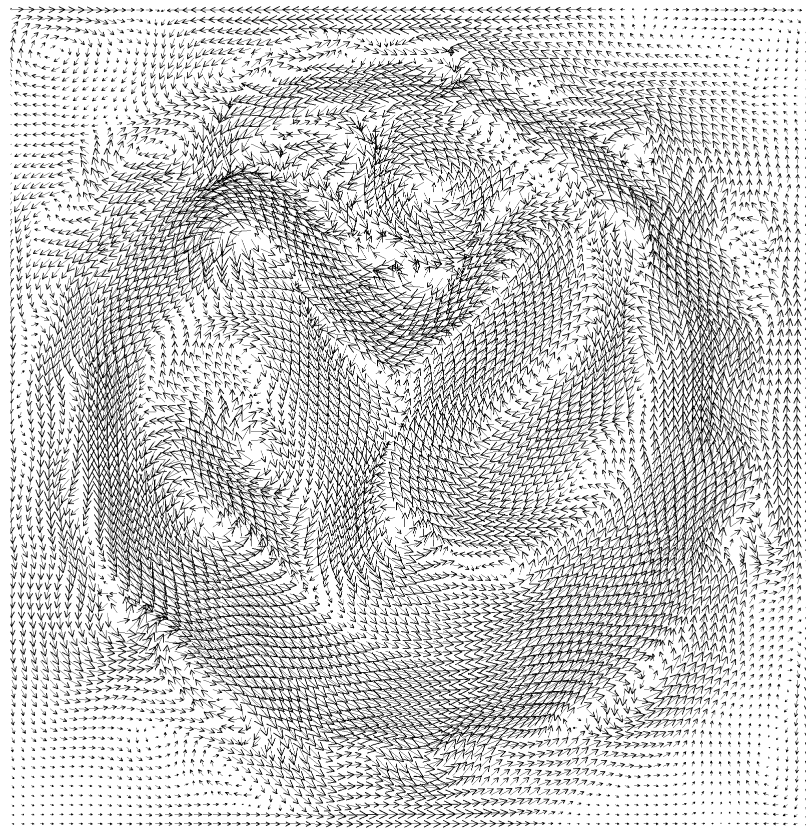


Potential vorticity at 20 days

Mass transport after 60 days

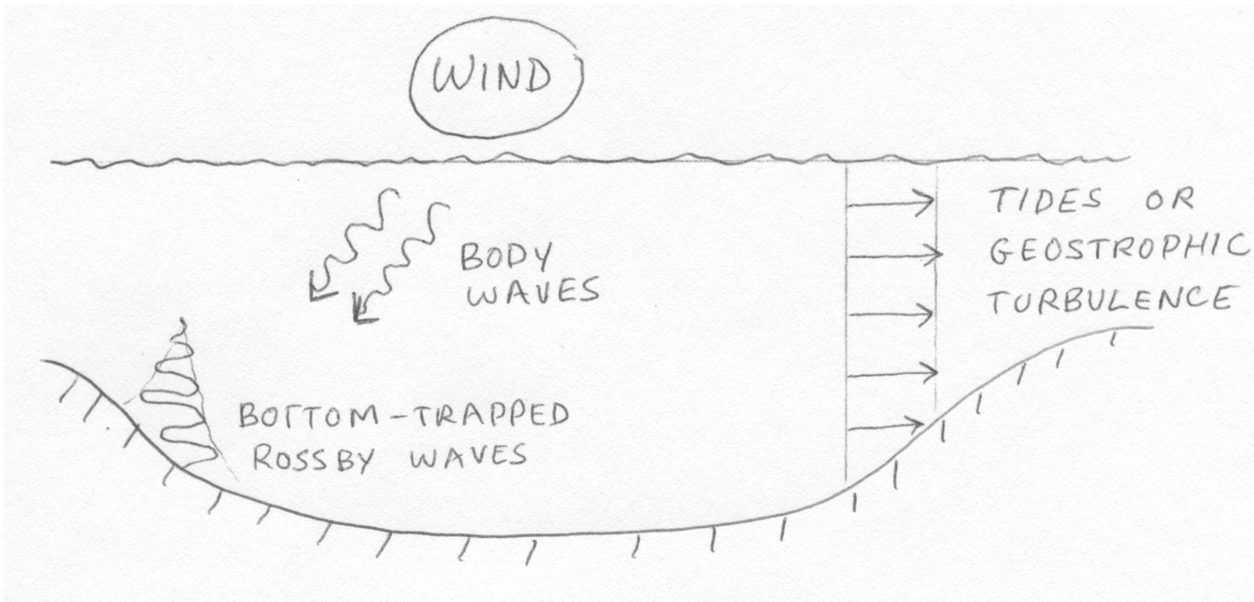


No ridge



Ridge

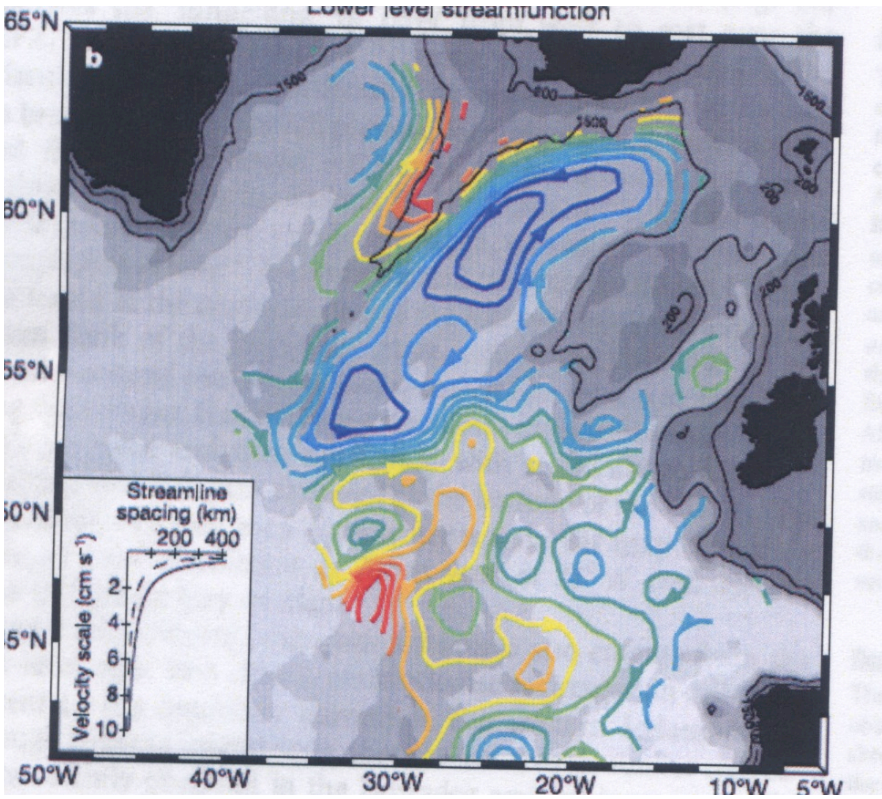
How this might work…



Think of the deep ocean as a single layer with an external forcing.

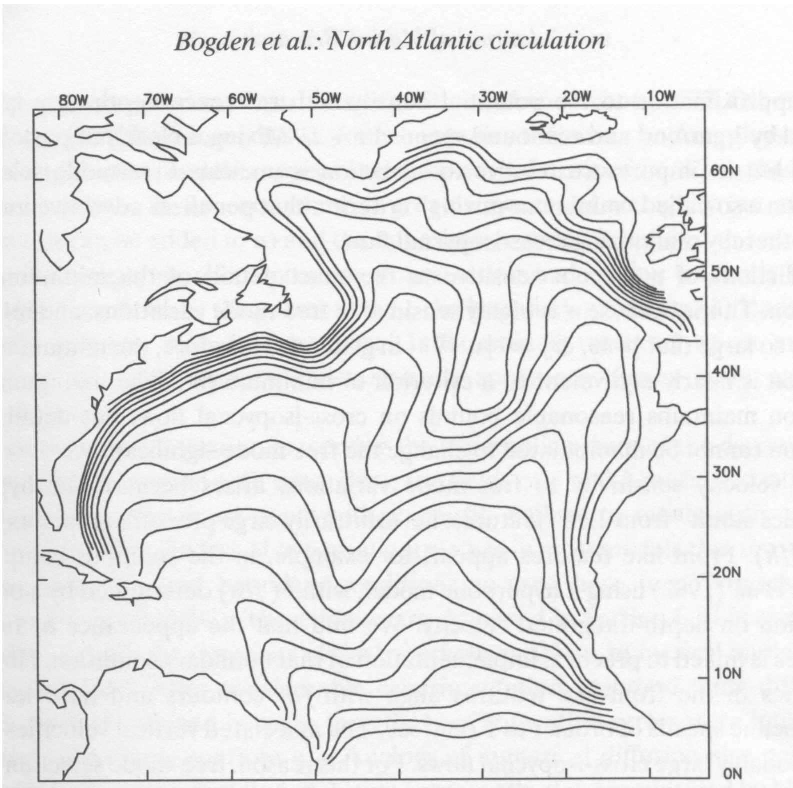
Mean streamfunction at 1500-1750 m in NE Atlantic

based on 223 floats. Bower et al., *Nature* 2002



Westward intensification

At the largest spatial scales the lines of constant *f*/*H* do not close.



f/H lines on the North Atlantic

In the limit of a flat bottom, they are lines of constant latitude.

Single-layer, flat bottom QG in a *basin*

For the flat-bottom case, $\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla(\zeta + f) = 0$

implies conservation of

$$\iint d\mathbf{x}\,(\zeta + f)^2 = \iint d\mathbf{x}\,\zeta^2 + \iint d\mathbf{x}\,\zeta\,\beta y + \cdots$$

Increasing $\iint d\mathbf{x}\,\zeta^2$ implies decreasing $\iint d\mathbf{x}\,\zeta\,\beta y$ (westward mean flow)

Moreover:

$$\frac{d}{dt}\iint d\mathbf{x}\,\zeta\,\beta y = -\iint d\mathbf{x}\,\nabla \cdot (\mathbf{v}\zeta)\,\beta y = +\iint d\mathbf{x}\,\mathbf{v}\zeta \cdot \nabla(\beta y)$$

$$= \iint d\mathbf{x}\,v\zeta\,\beta == \iint d\mathbf{x}\,v\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\beta = \beta\left[\int v^2\,dy\right]_{west}^{east}$$

Westward intensification of eddy activity

An alternative approach to all this:
The Statistical Mechanics Viewpoint

Answers the question: What happens if you stir the ocean up and then wait an infinitely long time?

Assumes:
1. Inviscid ocean
2. Truncated to finite degrees of freedom

For the present problem SM predicts:

1. Large-scale time-average flow that obeys:

$$\frac{\nabla \cdot \left(\frac{1}{H} \nabla \psi \right) + f}{H} = a\psi + b$$

2. Time fluctuations at the smallest scales

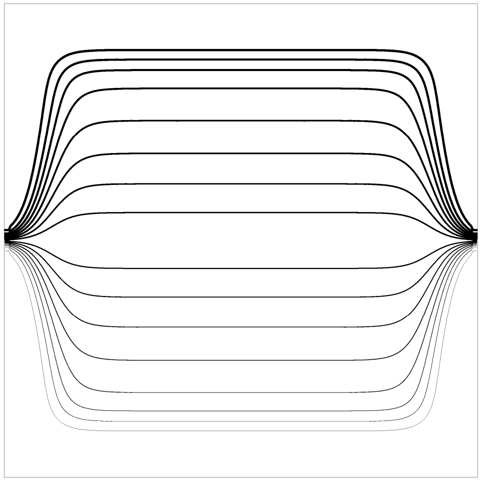
Best general reference: Carnevale & Frederiksen, *JFM* 1987

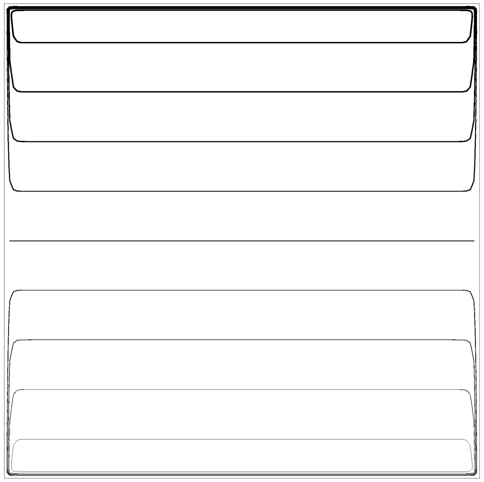
Recent application to this very problem:
Merryfield, Cummins & Holloway, *JPO* 2001

Maximum entropy flow states

$$\frac{\nabla \cdot \left(\frac{1}{H} \nabla \psi \right) + f}{H} = a\psi + b$$

f/H lines
with shelf/slope

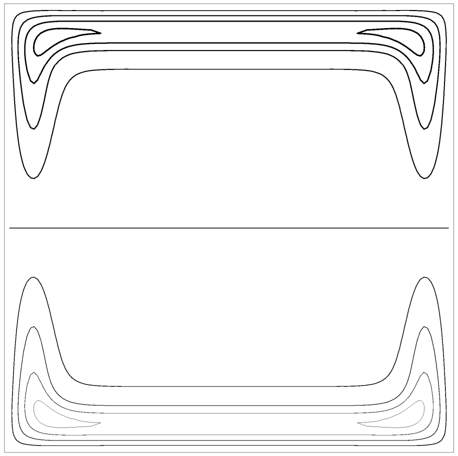




ψ

$f = \beta y, \quad H = 4\ km$

'Fofonoff flow'



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Flat-bottom, beta-plane case: Approach to ‘Fofonoff flow’

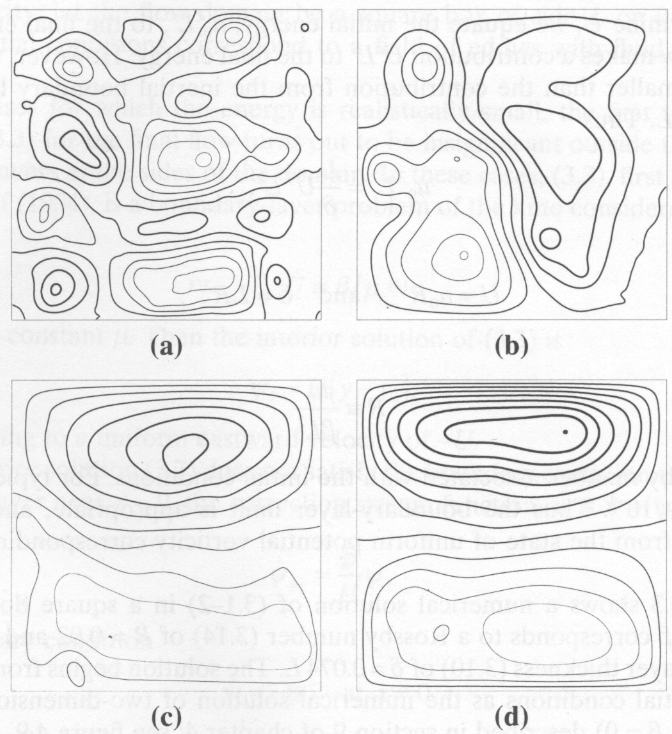
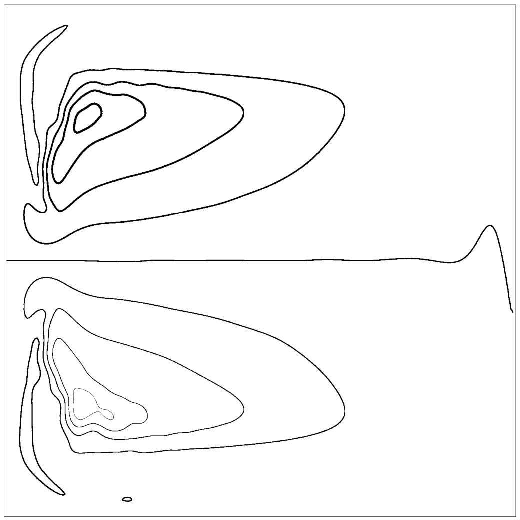


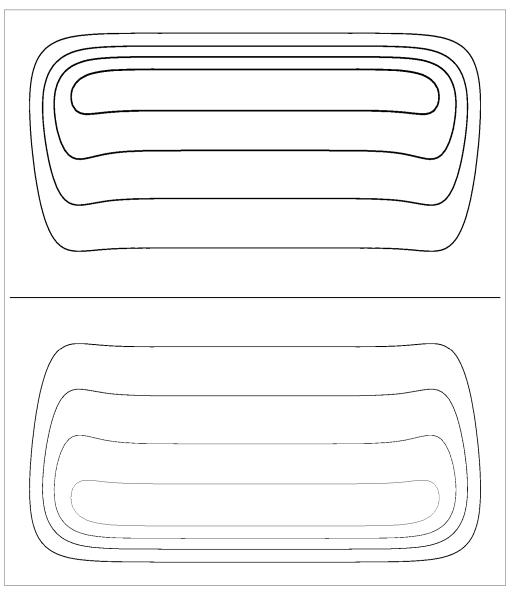
Figure 6.3 The stream function for quasigeostrophic flow in a bounded beta-plane box, according to a solution of (3.1–2) at times (a) $t = 1.0$, (b) $t = 3.0$, and (c) $t = 7.0$ after the random initial conditions of figure 4.9, and (d) the time-averaged stream function from $t = 5.0$ to 7.0. Darker contours correspond to larger values. The numerical solution is slowly approaching the state predicted by (3.11).

Wind-driven flow



East wind

West wind



Statistical-mechanical target state